Numerical Cosmology: Building a Dynamical Universe

David Garrison
University of Houston Clear Lake
The Beginning
Where the Heck did all that come from?
First Observatories
New Technologies
Putting it all together
Not Everyone Understands the Theory
We still don’t know how physics works in this era yet.

The observable universe

We know what’s going on based on our knowledge of plasma physics and elementary particle physics.

We still don’t know how the universe is going to end yet.

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Inflation and Gravitational Waves
What are Gravitational Waves?

Gravitational Waves first appeared as part of Einstein’s General Theory of Relativity.
Einstein’s Theory of General Relativity

- **Space-time** tells **matter** how to move
- **Matter** tells **space-time** how to curve

- **Gravitational Waves**: Ripples in the fabric of space-time
- **Black Holes**: The final fate in the collapse of matter
What Do Gravitational Waves Look Like?

- Plus Polarization
- Cross Polarization
How GW interferometers work

\[ \frac{\Delta L}{L} = \text{Wave Strength} \approx 10^{-21} \]

- If \( \Delta L = \frac{1}{10} \) millimeter,
  
  Need \( L \approx 3 \) light years

- If \( \Delta L = 10^{-13} \text{ cm} \) (Diameter of atomic nucleus),
  
  Need \( L \approx 4000 \) kilometers

- Can measure \( \Delta L = 10^{-16} \text{ cm} \times \frac{1}{1000} \times \) (Diameter of atomic nucleus),
  
  Build \( L = 4 \) kilometers
LISA Space-based Gravitational Wave Observatory
Low-Frequency Band: 0.1 to 0.0001 Hz

**LISA**
Laser Interferometer Space Antenna

$L = 5$ million kilometers
GW Spectrum
RMS Amplitude vs Frequency

Gravity Wave Spectrum - All Models - $\beta = -2$ & $-1.9$
Simplified BRF GW

Planck scale

$\Delta$ Standard vs BRF for $\beta = -2$

$\Delta$ Standard vs BRF for $\beta = -1.9$
Gravitational vs EM Radiation

<table>
<thead>
<tr>
<th>GRavitационAL WAVES CONTRASTED WITH ELECTROMAGNETIC WAVES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>Oscillations of EM field propagating through spacetime</td>
</tr>
<tr>
<td>Incoherent superposition of waves from molecules, atoms, and particles</td>
</tr>
<tr>
<td>Frequencies ~ 1 MHz and upward 20 orders</td>
</tr>
<tr>
<td>Easily absorbed and scattered</td>
</tr>
<tr>
<td>Emitted from surfaces of objects (where optically thin and gravity is weak)</td>
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</table>

**Implications:**

Gravitational waves are the ideal tool for probing strong-gravity regions of spacetime (general relativity)

Gravitational waves have the potential to bring us great surprises --- a “revolution” in our understanding of gravity and the Universe

Because of differences in EM and Gravitational Radiation, observing GWs is very different and so requires a different kind of astronomy.
Why We Care about GWs

• Gravitational Waves can excite (turbulent?) modes of oscillation in the plasma field like a crystal is excited by sound waves.
• What are the results of these excited modes? What part did they play in the evolution of the universe?
• Can these excited modes contribute to the formation of structures in the early universe?
Magnetohydrodynamic (Plasma) Turbulence

• Plasma (ionized gas): charged-particles or magneto-fluid

• Plasma kinetic theory – particle description: Probability Density Function (p.d.f.) $f_j(x, p, t)$, $j = e^-, ions$.

• MagnetoHydroDynamics (MHD) – $u(x, t)$, $B(x, t)$ and $p(x, t)$.

• MHD turbulence – $u$, $B$ and $p$ are random variables (mean & std. dev.).

• External magnetic fields & rotation affect plasma dynamics.
Homogeneous MHD Turbulence

 Examiner flow in a small 3-D cube (3-torus).

 Assume periodicity and use Fourier series.

 Homogeneous means same statistics at different positions.

 Approximation that focuses on physics of turbulence.

 Periodic cube is a surrogate for a compact magneto-fluid.
Fourier Analysis

Represent velocity and magnetic fields in terms of Fourier coefficients;

\[
\mathbf{u}(\mathbf{x}, t) = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{x}), \quad \mathbf{k} \cdot \tilde{\mathbf{u}}(\mathbf{k}, t) = 0
\]

\[
\mathbf{b}(\mathbf{x}, t) = \frac{1}{N^{3/2}} \sum_{\mathbf{k}} \tilde{\mathbf{b}}(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \mathbf{x}), \quad \mathbf{k} \cdot \tilde{\mathbf{b}}(\mathbf{k}, t) = 0
\]

Wave vector: \( \mathbf{k} = (n_x, n_y, n_z) \), where \( n_m \in \{..., -3, -2, -1, 0, 1, 2, 3, ... \} \)

Wave length: \( \lambda_k = 2\pi/|\mathbf{k}| \). Numerically, we use only \( 0 < |\mathbf{k}| \leq K \).

In computational physics, this is called a ‘spectral method’.
Fourier-Transformed MHD Equations

Below, $Q_u$ and $Q_b$ are nonlinear terms involving products of the velocity and magnetic field coefficients. In “$k$-space”, we have

$$\frac{d\tilde{u}(k)}{dt} = Q_u(k) + 2\tilde{u}(k)\times\Omega + ik \cdot B_0 \tilde{b}(k) - \nu k^2 \tilde{u}(k)$$

$$\frac{d\tilde{b}(k)}{dt} = Q_b(k) + ik \cdot B_0 \tilde{u}(k) - \eta k^2 \tilde{b}(k).$$

Direct numerical simulation (DNS) includes $\mathcal{N}$ modes with $k$ such that $0 < |k| \leq k_{max}$ and so defines a dynamical system of independent Fourier modes.
Non-linear Terms

The $Q_u$ and $Q_b$ are convolution sums in $k$-space:

$$Q_u (k) = \left( \mathbb{I} - \hat{k}\hat{k} \right) \cdot \sum_{p+q=k} \left[ \tilde{\mathbf{u}}(p) \times \tilde{\mathbf{w}}(q) + \tilde{\mathbf{j}}(p) \times \tilde{\mathbf{b}}(q) \right]$$

$$Q_b (k) = i\mathbf{k} \times \sum_{p+q=k} \tilde{\mathbf{u}}(p) \times \tilde{\mathbf{b}}(q)$$

$$\tilde{\mathbf{w}}(q) = i\mathbf{q} \times \tilde{\mathbf{u}}(q), \quad \tilde{\mathbf{j}}(p) = i\mathbf{p} \times \tilde{\mathbf{b}}(p).$$

Since $\nabla_k \cdot Q_u (k) = \nabla_k \cdot Q_b (k) = 0$, ideal MHD flows satisfy a Liouville theorem.
Statistical Mechanics of MHD Turbulence

- ‘Atoms’ are components of Fourier modes $\tilde{u}(k), b(k)$.
- Canonical ensembles can be used (T.D. Lee, 1952).
- Gases have one invariant, the energy $E$.
- **Ideal MHD** ($\nu = \eta = 0$) has $E, H_C$ and $H_M$.
- $H_C$ and $H_M$ are pseudoscalars under $P$ or $C$ or both.
- Ideal MHD statistics exists, but not same as $\nu, \eta \to 0+$.
- However, low-$k$ ideal & real dynamics may be similar.
### Ideal Invariants with $\Omega_0$ and $B_0$

3-D MHD Turbulence, with $\Omega_0$ and $B_0$ has various ideal invariants:

<table>
<thead>
<tr>
<th>Case</th>
<th>Mean Field</th>
<th>Angular Velocity</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td>0</td>
<td>$E, H_C, H_M$</td>
</tr>
<tr>
<td>II</td>
<td>$B_0 \neq 0$</td>
<td>0</td>
<td>$E, H_C$</td>
</tr>
<tr>
<td>III</td>
<td>0</td>
<td>$\Omega_0 \neq 0$</td>
<td>$E, H_M$</td>
</tr>
<tr>
<td>IV</td>
<td>$B_0 \neq 0$</td>
<td>$\Omega_0 = \sigma B_0$</td>
<td>$E, H_P$</td>
</tr>
<tr>
<td>V</td>
<td>$B_0 \neq 0$</td>
<td>$\Omega_0 \neq 0 (B_0 \times \Omega_0 \neq 0)$</td>
<td>$E$</td>
</tr>
</tbody>
</table>

In Case V, the ‘parallel helicity’ is $H_P = H_C - \sigma H_M$ ($\sigma = \Omega_0 / B_0$).
Statistical Mechanics of Ideal MHD

\[ E = \frac{1}{2N^3} \sum_k \left[ |\tilde{u}(k)|^2 + |\tilde{b}(k)|^2 \right] \]

Ideal invariants:
\[ H_C = \frac{1}{2N^3} \sum_k \tilde{u}(k) \cdot \tilde{b}^*(k) \]
\[ H_M = \frac{1}{2N^3} \sum_k \frac{i}{k^2} k \cdot \tilde{b}(k) \times \tilde{b}^*(k) \]

Phase Space Probability Density Function:
\[ D = Z^{-1} \exp(-\alpha E - \beta H_C - \gamma H_M) = Z^{-1} \exp(-\Sigma_k y^\dagger M y) \]

\( \alpha, \beta, \gamma \) are `inverse temperatures’; \( y^\dagger = (u_1,u_2,b_1,b_2) \)

\( \beta, \gamma, H_C, H_M \) are pseudoscalars under \( P \) and \( C \).
Eigenvariables

There is a unitary transformation in phase space such that

\[ [u_1(k), u_2(k), b_1(k), b_2(k)] \rightarrow [v_1(k), v_2(k), v_3(k), v_4(k)] \]

\[
D = \prod_k D(k) = \prod_k \frac{1}{Z(k)} \exp \left( -\frac{1}{N^3} \sum_{j=1}^{4} \lambda_k^{(j)} |v_j(k)|^2 \right)
\]

The \(v_j(k)\) are eigenvariables and the \(\lambda_k^{(j)}\) are eigenvalues of the unitary transformation matrix.
Phase Portraits

Although the dimension of phase space may be $\sim 10^6$, and the dynamics of the system is represented by a point moving on a trajectory in this space, we can project the trajectory onto 2-D planes to see it:
Coherent Structure, Case III (Rotating)

\[ \alpha = 1.01862, \quad \beta = 0.00000, \quad \gamma = -1.017937 \]

**Non-ergodicity** indicated by large mean values: time-averages \( \neq \) ensemble averages.

**Birkhoff-Khinchin Theorem**: non-ergodicity = surface of constant energy disjoint.

**Surface of constant energy** is disjoint in ideal, homogeneous MHD turbulence.
Coherent magnetic energy density in the $z = 15$ plane of a $32^3$ simulation (averaged from $t=0$ to $t=1000$)
The Goal of This Work

- Apply the physics / mathematics of MHD Turbulence to Gravitational Waves / Relativistic Plasmas
- Demonstrate the formation of coherent structures (cosmic magnetic fields, density and temperature variations and relic gravitational waves) as a result of interactions with gravitational waves
- Utilize a GRMHD code to model both the plasma and the background space-time dynamically
- Study the interaction between MHD turbulence and gravitational waves and vice-versa
Our Approach

• Simulate the early universe after the inflationary event when the universe was populated by only a Homogeneous Plasma Field and Gravitational Radiation generated by inflation
• At this stage “classical” physics, General Relativity and Magneto-hydrodynamics, can describe the evolution of the universe
• We start with initial conditions at $t = 3$ min and evolve these conditions numerically using the GRMHD equations
GRMHD Variables - Spacetime

- **Spacetime metric:**
  \[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij} (\vec{x}, t) (dx^i + \beta^i dt)(dx^j + \beta^j dt)
\]

- **Extrinsic Curvature:**
  \[
  \mathbf{K}_{ij} = -\frac{1}{2\alpha} (\partial_t - L_\beta) \gamma_{ij} (\vec{x}, t)
  \]

- **BSSN Evolution Variables:**
  \[
  \phi = \frac{1}{12} \ln[\det(\gamma_{ij})]
  \]
  \[
  \tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}
  \]
  \[
  K = \gamma^{ij} K_{ij}
  \]
  \[
  \tilde{A}_{ij} = e^{-4\phi} (K_{ij} - \frac{1}{3} \gamma_{ij} K)
  \]
  \[
  \tilde{\Gamma}^i = -\tilde{\gamma}^{ij},_j
  \]
GRMHD Variables - MHD

\[ \rho_* = \alpha \sqrt{\gamma \rho_0 u^0} : \text{conserved mass density} \]

\[ S_i = \alpha \sqrt{\gamma T_i^0} : \text{momentum density} \]

\[ \tau = \alpha^2 \sqrt{\gamma T^{00}} - \rho_* : \text{energy density} \]

\[ \tilde{B}^j = \sqrt{\gamma B^j} : \text{magnetic field} \]

\[ v^i = \frac{1}{u^0} \gamma^{ij} u_j - \beta^i : 3 - \text{velocity} \]

\[ u^0 = \frac{1}{\alpha} \sqrt{1 + \gamma^{ij} u_i u_j} \]

\[ P = (\Gamma - 1) \rho_0 \varepsilon : \text{pressure} \]
## Stress-Energy Tensor

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu} \] Einstein's Eqn

\[ T^{\mu\nu} = (\rho_0 h + b^2) u^\mu u^\nu + (P + \frac{b^2}{2}) g^{\mu\nu} - b^\mu b^\nu \]

\[ h = 1 + \varepsilon + \frac{P}{\rho_0} \] Enthalpy

\[ b^\mu = \frac{1}{\sqrt{4\pi}} B_{(u)}^\mu \]

\[ B_{(u)}^0 = \frac{1}{\alpha} u_i B^i \quad ; \quad B_{(u)}^i = \frac{1}{u^0} \left( \frac{B^i}{\alpha} + B_{(u)}^0 u^i \right) \]
Building our Model

- The observer is co-moving with fluid therefore $\alpha = 1, \beta = 0, u^i = (1, 0, 0, 0)$
- Beginning of Classical Plasma Phase, $t = 3$ min
- $T = 10^9$ K, Plasma is composed of electrons, protons, neutrons, neutrinos and photons
- Mass-Energy density is $10^4$ kg/m$^3$
- The universe is radiation-dominated
- The Hubble parameter at this time is $7.6 \times 10^{16}$ km/s/Mpc
Other Parameters

• Age of the Universe 13.7 Billion Years
• Scale Factor: \( a(3.0 \text{ min}) = 2.81 \times 10^{-9} \)
• Specific Internal Energy, \( \varepsilon \) calculated from \( T \)
• Pressure, \( P \): calculated using the Gamma Law with \( \Gamma = 4/3 \)
• The Electric Field is set to zero b/c the observer is co-moving with the fluid
• The Magnetic field is set to \( 10^{-9} \) G based on theoretical estimates of the primordial seed field
Initial Spacetime

- Perturbed Robertson-Walker Metric

\[ ds^2 = a(t)^2[-dt^2 + (\delta_{ij} + h_{ij})dx^i \, dx^j] \]

- Spectrum of Perturbations

\[ h(k, t) = 8\sqrt{\pi} l_{pl} |1 + \chi|^{-(1+\chi)}k^{2+\chi} / l_0 \]

- Birefriengence

\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu h_{ij}^L) = -2i \frac{\theta}{a} \dot{h}_{ij}^L \]

\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu h_{ij}^R) = +2i \frac{\theta}{a} \dot{h}_{ij}^R \]
Preliminary Results

Density Variation vs. Time

Seconds after $t = 3$ minutes
Future Developments

• Rewrite GR and GRMHD Equations in k-space so we can use spectral methods
• Add Viscosity
• Add Scalar Metric Perturbations
• Add Scalar Fields if needed
• Incorporate a Logarithmic Computational Grid
Questions?